Topological Representations of $U_q(sl_2(\mathbb{C}))$ on the Torus and the Mapping Class Group

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Abstract: We compute the mapping class group action on cycles on the configuration space of the torus with one puncture, with coefficients in a local system arising in conformal field theory. This action commutes with the topological action of the quantum group $U_q(sl_2(\mathbb{C}))$, and is given in vertex form.

1. Introduction

We consider topological representations of $U_q(sl_2(\mathbb{C}))$ appearing in free field representations of conformal field theories on the torus based on $SU(2)$. Topological representations of quantum groups on the complex plane were introduced in [1,2,3]. The torus has been investigated in [4].

Fock space traces of products of vertex operators yield multivalued holomorphic differential forms on configuration spaces over the torus. Quantum groups [5-8] enter through their action on a certain space A of linear forms on the space of holomorphic multivalued differential forms with given monodromy. These forms are given by integration on products of loops. Singular vectors with respect with this action give cycles, and define thus linear forms on cohomology. We consider the torus with one puncture together with the local system given in [4], associated to the monodromy of the differential forms. We restrict our attention to the quantum group $U_q(sl_2(\mathbb{C}))$ at q a $2p$'th root of unity. The topological action of $U_q(sl_2(\mathbb{C}))$ has been identified in [4] with the adjoint representation in the sense that the space A is isomorphic to $U_q(sl_2(\mathbb{C}))$ as $U_q(sl_2(\mathbb{C}))$-module with the adjoint action. The main input from conformal field theory [9-17] is the form of the local system.

An important feature of conformal field theories on the torus is modular invariance [17]. A natural question to pose is the meaning of modular transformations on the side of topological representations. The first observation is that the local system coming from conformal field theory is compatible with modular transformations in a sense to be defined below. As a consequence the modular group acts on the space A. The second observation is that we can explicitly compute the action of the modular group on A by contour deformation methods. Using the identification of A with the quantum group algebra, we obtain the action of the modular group on the latter.

Since the action of the modular group commutes with the topological action of $U_q(sl_2(\mathbb{C}))$, it commutes on the algebraic side with the adjoint action. The result is a "vertex" form of the generators T and S of the modular group. These generators are expressed in terms of the R-matrix of an enlarged version of $U_q(sl_2(\mathbb{C}))$ (the "K-generated algebra") and the Haar measure on $U_q(sl_2(\mathbb{C}))$. A representation of the mapping class group in "SOS" form arises in the study of three-manifold invariants of Reshetikhin and Turaev [8].

We obtain as a byproduct the quantum group interpretation of modular invariance. Namely, the action of the modular group leaves invariant the subspace of singular vectors in the adjoint representation.

It turned out that very similar formulas have been discovered independently, in the context of braided groups of monoidal categories, by Lyubashenko and Majid [18].

2. Configuration spaces and local systems on the torus

Let X be a torus with one puncture.

2.1. Representation of X

Let $D_R(w) = \{z \in \mathbb{C} | |z - w| < r\}$, the open disc. We represent $X = (\overline{D_R(0)} \backslash \cup_{i=1}^{2} D_{R'}(w_i))/\sim$, the disc with two holes, the boundaries of which we identify. E.g., we take $w_1 = -\frac{R}{3}$ and $w_2 = \frac{R}{3}$, define $\phi(w_1 + R' e^{i(\varphi - \pi)}) = w_2 + R' e^{-i\varphi}$, and identify $\phi(z) \sim z$. Let $p_0 = -R$ serve as a base point. $\pi_1(X, p_0)$ is generated by elements α and β as represented by the loops in X based at p_0 shown in figure (1). For later purpose, we introduce the abbreviations $\alpha' = \beta^{-1} \circ \alpha^{-1} \circ \beta$ and $\gamma = \alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$.

2.2. Configuration spaces and braid groups on X

For $r \geq 1$, we define configuration spaces

$$X_r = \left(X^r \setminus \cup_{1 \leq i < j \leq r} \{(z_1, \ldots, z_r) \in X^r | z_i = z_j\}\right) / S_r. \tag{2.1}$$

S_r is the symmetric group, acting from the right. Let $*_r = [x_1, \ldots, x_r]$ be a base point in X_r. $\pi_1(X_r, *_r)$ is the braid group with r strings on X. We call a base point admissible if $-R \leq x_1 < \ldots < x_r \leq -\frac{R}{3} - R'$. Braid groups defined with respect to different admissible base points are canonically isomorphic. We will always assume $*_r$ to be an admissible base point such that $\{x_1, \ldots, x_r\} \subset \partial X$. $\pi_1(X_r, *_r)$ is generated by elements σ_i, $1 \leq i \leq r-1$, α, and β. Intuitively, σ_i interchanges x_i with x_{i+1} counterclockwise, while α and β move x_r along the respective loops, other components of $*_r$ being kept fixed. An abundance of relations hold among these generators. We will not present them here.

2.3. Local systems over X_r

Let p be an odd positive integer. Put $q = e^{\frac{\pi i}{p}}$ and define $2p$ by $2p$ matrices A and B with entries

$$\begin{aligned} A_{m,n} &= q^{1-m} \delta_{m,n}, \\ B_{m,n} &= \sum_{l \in \mathbb{Z}} \delta_{m,n+2pl+1}. \end{aligned} \tag{2.2}$$

They satisfy $AB = q^{-1} BA$. Let $V = \mathbb{C}^{2p}$. The assignments $\rho_r(\sigma_i) = -q^2$, $1 \leq i \leq r-1$, $\rho_r(\alpha) = A^2$, and $\rho_r(\beta) = B^2$ define a $2p$-dimensional representation $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$. It is the monodromy representation associated with multivalued differential forms on X_r mentioned above. ρ_r is the direct sum of two equivalent p-dimensional irreducible representations.

Let X_r^0 be the subspace of X_r consisting of configurations which contain p_0 among their components. We then define $\phi_r : X_{r-1} \setminus X_{r-1}^0 \to X_r^0$ to be the bijection which inserts p_0. The family of representations ρ_r, $r \geq 1$, is compatible in the following sense. Let $\pi_1(\phi_r) : \pi_1(X_{r-1} \setminus X^0_{r-1}, *_{r-1}) \to \pi_1(X^0_r, \phi_r(*_{r-1}))$ be the isomorphism induced by ϕ_r, then $\rho_r \circ \pi_1(\phi_r) = \rho_{r-1}$.

With ρ_r we associate the local system $L_r(X) = \hat{X}_r(*_r) \otimes_{\pi_1(X_r,*_r)} V$, a flat vector bundle over X_r with distinguished trivialization over $*_r$, the holonomy associated with elements of $\pi_1(X_r, *_r)$ being ρ_r. Due to the compatibility, ϕ_r can be lifted to $L_r(\phi_r) : L_{r-1}(X)|_{X_{r-1}\setminus X^0_{r-1}} \to L_r(X)|_{X^0_r}$. We define $L_r(\phi_r)([x, v]) = [\phi_r(x), v]$, which is checked to be well defined.

3. Topological representations of $U_q^{red}(sl_2(\mathbb{C}))$

We summarize briefly the constructions leading to topological representations of $U_q^{red}(sl_2(\mathbb{C}))$ adjusting the notations to the present setup.

3.1. Families of nonintersecting loops with values in the local system

Let $Q_r =]0,1[^r \cup \bigcup_{i=1}^r]0,1[\times \ldots \times \{0,1\} \times \ldots \times]0,1[$ and $\gamma_1, \ldots, \gamma_r$ be loops $[0,1] \to X$ starting and ending at p_0, nonintersecting except at p_0. Define $[\gamma_1, \ldots, \gamma_r] : Q_r \to X_r$ be the corresponding embedding. Denote by $[\beta^j, \alpha^k] : Q_r \to X_r$, $j + k = r$ a family of nonintersecting loops obtained by homotopic deformation of j β-loops and k α-loops given by

$$[\beta^j, \alpha^k](t_0, \ldots, t_{r-1}) = [\beta^{(0)}(t_0), \ldots, \beta^{(j-1)}(t_{j-1}), \alpha^{(j)}(t_j), \ldots, \alpha^{(r-1)}(t_{r-1})]. \tag{3.1}$$

It represents a locally finite r-chain in X_r with boundary in X^0_r.

We lift it to take values in $\hat{X}_r(*_r)$. We specify the lift by choosing an admissible point on its image, connecting this point to the base point by an admissible path. Then the equivalence class $[[\beta^j, \alpha^k], v]$, $v \in V$, defines a family of nonintersecting loops in X with values in $L_r(X)$. The space of families of nonintersecting loops in X with values in $L_r(X)$ is denoted by $A_r(X_r, X^0_r; L_r)$ or shorter by A_r. Its precise definition contains equivalence relations reflecting the possibility of homotopic deformation, reparametrization, and splitting of loops (see [1]). The elements $[[\beta^j, \alpha^k], e_n]$, $0 \le j, k \le \min\{r, p-1\}$ such that $j + k = r$, and $1 \le n \le 2p$, constitute a basis. Here $(e_n)_m = \delta_{n,m}$. A family which contains p homotopic loops is put equivalent to zero. Therefore, we restrict ourselfs to $r \le 2p - 2$.

3.2. Topological action of $U_q^{red}(sl_2(\mathbb{C}))$

The basic ingredience of topological representations are operators $E : A_r \to A_{r-1}$, $F : A_r \to A_{r+1}$, and $K^2 : A_r \to A_r$ defined by

$$\begin{aligned} E[[\beta^j, \alpha^k], e_n] &= -L_r(\phi_r)^{-1}\partial[[\beta^j, \alpha^k], e_n], \\ F[[\beta^j, \alpha^k], e_n] &= q^{-2j-2k-2}[[\beta^j, \alpha^k, \gamma], e_n], \\ K^2[[\beta^j, \alpha^k], e_n] &= q^{-2r-2}[[\beta^j, \alpha^k], e_n]. \end{aligned} \tag{3.2}$$

They are shown to satisfy the relations of $U_q(sl_2(\mathbb{C}))$: $K^2E = q^2EK^2$,$K^2F = q^{-2}FK^2$, $[E, F] = K^2 - K^{-2}$, and the additional relations $E^p = F^p = (K^2)^{2p} - 1 = 0$, defining $U_q^{red}(sl_2(\mathbb{C}))$.

Thus $\bigoplus_{r=0}^{2p-2} A_r$ comes equipped with the structure of a module over $U_q^{red}(sl_2(\mathbb{C}))$. We identify this representation as the adjoint representation. Let $\phi : \bigoplus_{r=0}^{2p-2} A_r \to U_q^{red}(sl_2(\mathbb{C}))$ be the map

$$\begin{aligned} &\phi[[\beta^j, \alpha^k], e_n] = N(j, k, n)F^kT_{n-1}E^{p-1-j}, \\ &\qquad N(j, k, n) = (-1)^j q^{2j+2n} q^{\frac{1}{2}j(j-1)} \frac{[j]_q!}{[1]_q^j} q^{(j+k)(j+k-1)+(j+k)(1-n)+j(1+n)}. \end{aligned} \tag{3.3}$$

An explicit computation proves that ϕ is a homomorphism of $U_q^{red}(sl_2(\mathbb{C}))$ modules. Moreover, it is one-to-one and onto. Here $T_n = \frac{1}{2p}\sum_{m=0}^{2p-1} q^{-nm}K^{2m}$.

The actions of $U_q^{red}(sl_2(\mathbb{C}))$ on itself by left multiplication and by right multiplication, twisted with the antipode, also have topological counterparts. The operator which implements left multiplication by F is called F_L. The operator which corresponds to right multiplication by $\eta(F)$ is denoted by F_R. On the topological side they are given by

$$\begin{aligned} F_L[[\beta^j, \alpha^k], e_n] &= q^{n-1-2j-2k}[[\beta^j, \alpha^{k+1}], e_n], \\ F_R[[\beta^j, \alpha^k], e_n] &= q^{-2j-2k-2}[[\alpha', \beta^j, \alpha^k], e_n]. \end{aligned} \tag{3.4}$$

Intuitively, F adds a γ-loop, while F_L and F_R add α- and α'-loops respectively. The interpretation of $\bigoplus_{r=0}^{2p-2} A_r$ as a bimodule will not be worked out here.

The important formula of this section to keep in mind is (3.3).

4. Action of the Mapping Class Group

Let Diff(X) be the group of diffeomorphisms which leave $\partial X = \{z \in \mathbb{C} \,|\, |z| = R\}$ invariant. Let Diff$_0(X)$ be the subgroup of Diff(X) consisting of diffeomorphisms homotopic to the identity. The mapping class group of X is defined as

$$\mathcal{M}_{1,1}(X) := \mathrm{Diff}(X)/\mathrm{Diff}_0(X). \tag{4.1}$$

A reference on mapping class groups is [18].

4.1. Generators of $\mathcal{M}_{1,1}(X)$

$\mathcal{M}_{1,1}(X)$ is generated by Dehn twists. On the torus, we have two kinds of Dehn twists, T_α and T_β. T_α is defined as follows. Consider the annulus $\{z \in X \mid r+\epsilon \leq |z-w_1| \leq r+2\epsilon\}$ with $\epsilon > 0$ fixed. We define a map of this annulus to itself by $T_\alpha(w_1+z) := w_1 + e^{i\varphi(|z|)}z$ with φ a smooth function interpolating between $\varphi(r+\epsilon) = 0$ and $\varphi(r+2\epsilon) = 2\pi$. We say that T_α is the Dehn twist associated with the loop $t \mapsto w_1 - (r+2\epsilon)e^{2\pi it}$, $t \in [0,1]$. The Dehn twist T_β is associated with the loop $t \mapsto (w_1+r)(1-t)+(w_2-r)t$. See figure (3). The orientations we use are shown by arrows. T_α and T_β leave the base point $*_r$ invariant. (Recall that $*_r$ is a configuration on ∂X.) Thus we have a map from $\mathcal{M}_{1,1}(X)$ to $\mathrm{Aut}(\pi_1(X_r, *_r))$, the automorphisms of $\pi_1(X_r, *_r)$, $r \geq 1$.

4.2. Compatibility of local systems

We define a representation $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$ to be compatible with $\mathcal{M}_{1,1}(X)$ if $\rho \circ T \simeq \rho$ for all $T \in \mathcal{M}_{1,1}(X)$. This means that for every T there exists a matrix $D(T) \in \mathrm{GL}(V)$ such that

$$\rho \circ T(\sigma) = D(T)\rho(\sigma)D(T)^{-1}. \tag{4.2}$$

If a representation ρ_r is compatible with $\mathcal{M}_{1,1}(X)$, then we have an action of $\mathcal{M}_{1,1}(X)$ on L_r given by

$$L_r(T) : L_r \to L_r,\ [x,v] \mapsto [T(x), D(T)v], \tag{4.3}$$

which is well defined due to equation (4.2).

The family of local systems $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$, $r \geq 1$, is indeed compatible with $\mathcal{M}_{1,1}(X)$. For the generators T_α and T_β the compatibility is proved by defining

$$\begin{aligned} D(T_\alpha) &:= \sum_{l=0}^{2p-1} q^{\frac{1}{2}l(l-2)} A^l, \\ D(T_\beta) &:= \sum_{l=0}^{2p-1} q^{-\frac{1}{2}l(l+2)} B^l. \end{aligned} \tag{4.4}$$

and by noting that the action of T_α has the form $T_\alpha(\alpha) = \alpha$ and $T_\alpha(\beta) = \alpha \circ \beta$, while that of T_β has the form $T_\beta(\alpha) = \alpha \circ \beta$ and $T_\beta(\beta) = \beta$. Note also that $D(T_\alpha)$ and $D(T_\beta)$ have matrix elements

$$
\begin{aligned}
D(T_\alpha)_{m,n} &= \left\{ \sum_{l=0}^{2p-1} q^{\frac{1}{2}l^2} \right\} q^{-\frac{1}{2}m^2} \delta_{m,n}, \\
D(T_\beta)_{m,n} &= \sum_{k \in \mathbb{Z}} \sum_{l=0}^{2p-1} q^{-\frac{1}{2}l(l+2)} \delta_{m,n+2pk+l}.
\end{aligned}
\tag{4.5}
$$

4.3. Action of $\mathcal{M}_{1,1}$ on $A_r(X_r, X_r^0; L_r)$

The mapping class group $\mathcal{M}_{1,1}$ acts on A_r as follows: $T \in \mathcal{M}_{1,1}$ acts by

$$
L_r(T)[[\gamma_1, \ldots, \gamma_r], v] = [T \circ [\gamma_1, \ldots, \gamma_r], D(T)v]. \tag{4.6}
$$

We compute the action of T_α and T_β on the basis elements $[[\beta^j, \alpha^k], e_n]$ of $A_r(X_r, X_r^0; L_r)$.

Action of T_α

Let us define $\delta := \alpha \circ \beta$. Let $r = j + k$. The action of T_α on $[[\beta^j, \alpha^k], e_n]$ is seen to have the form

$$
L_r(T_\alpha)([[\beta^j, \alpha^k], e_n]) = [[\delta^j, \alpha^k], D(T_\alpha)e_n]. \tag{4.7}
$$

The first problem is to decompose (4.7) in terms of the basis of $A_r(X_r, X_r^0; L_r)$. The decomposition is performed with the help of

$$
\begin{aligned}
&[[\beta^j, \delta^{l+1}, \alpha^k], e_n] \\
&\quad = [[\beta^{(0)}, \ldots, \beta^{(j-1)}, \delta^{(j)}, \ldots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \ldots, \alpha^{(j+l+k)}], e_n] \\
&\quad = [[\beta^{(0)}, \ldots, \beta^{(j)}, \delta^{(j+1)}, \ldots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \ldots, \alpha^{(j+l+k)}], e_n] \\
&\qquad + [[\beta^{(0)}, \ldots, \beta^{(j-1)}, \delta^{(j+1)}, \ldots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \ldots, \alpha^{(j+l+k)}, \alpha^{(j)}]\sigma, e_n] \\
&\quad = [[\beta^{j+1}, \delta^l, \alpha^k], e_n] + q^{2(l+k)}[[\beta^j, \delta^l, \alpha^{k+1}], e_{n+2}].
\end{aligned}
\tag{4.8}
$$

We use $\rho_{j+k+l+1}(\sigma) = (-q)^{2(k+l)}B^2$ and absorb $(-1)^{l+k}$ by an isometry of $Q_{j+l+k+1}$. The ordering of loops according to deformation and the assignement of the components of $(t_0, \ldots, t_{j+k+l}) \in Q_{j+l+k+1}$ to the individual loops indicated by superscripts should be clear from the notation used in (4.8). Iterate (4.8) to obtain

$$[[\delta^j,\alpha^k],e_n] = \sum_{s=0}^{\min\{j,p-k-1\}} c_s(j,k)[[\beta^{j-s},\alpha^{k+s}],e_{n+2s}] \tag{4.9}$$

with coefficients

$$c_s(j,k) = \sum_{0\le i_1\le\cdots\le i_s\le j-s}\prod_{l=1}^{s} q^{2(k+j-i_l-1)} = q^{s(j+2k+s-2)}\begin{bmatrix} j \\ s \end{bmatrix}_q \tag{4.10}$$

using Gauß's formula

$$\sum_{0\le i_1\le\cdots\le i_s\le j-s} q^{-2\sum_{l=1}^{s} i_l} = q^{s(s-j)}\begin{bmatrix} j \\ s \end{bmatrix}_q . \tag{4.11}$$

The q-binomial coefficient is defined as

$$\begin{bmatrix} j \\ s \end{bmatrix}_q := \frac{[j]_q!}{[s]_q![j-s]_q!}. \tag{4.12}$$

Putting (4.5),(4.7) , and (4.9) together, it follows that

$$L_r(T_\alpha)([[\beta^j,\alpha^k],e_n]) = dq^{-\frac{1}{2}n^2}\sum_{s=0}^{\min\{j,p-k-1\}} q^{s(j+2k+s-2)}\begin{bmatrix} j \\ s \end{bmatrix}_q [[\beta^{j-s},\alpha^{k+s}],e_{n+2s}] \tag{4.13}$$

with

$$d = \sum_{l=0}^{2p-1} q^{\frac{1}{2}l^2}. \tag{4.14}$$

Thus (4.13) gives the matrix elements of $L_r(T_\alpha)$ in terms of the basis with elements $[[\beta^j,\alpha^k],e_n]$.

Action of T_β

Let $r = j+k$. The action of T_β has the form

$$L_r(T_\beta)([[\beta^j,\alpha^k],e_n]) = [[\beta^j,\delta^k],D(T_\beta)e_n], \tag{4.15}$$

which we again will express in terms of the basis of $A_r(X_r, X_r^0; L_r)$. Using (4.8) it follows that

$$[[\beta^j, \delta^k], e_n] = \sum_{s=\max\{0,j+k-p+1\}}^{k} b_s(j,k)[[\beta^{j+k-s}, \alpha^s], e_{n+2s}] \tag{4.16}$$

with coefficients

$$b_s(j,k) = \sum_{0\le i_1\le\cdots\le i_s\le k-s} \prod_{l=1}^{s} q^{2(k-i_l-1)} = q^{s(k+s-2)} \begin{bmatrix} k \\ s \end{bmatrix}_q . \tag{4.17}$$

Note that $b_s(j,k) = c_s(k,0)$. (4.15),(4.16) , and (4.17) yield

$$\begin{aligned} &L_r(T_\beta)([\beta^j, \alpha^k], e_n]) \\ &\quad = \sum_{l=0}^{2p-1} \sum_{s=\max\{0,j+k-p+1\}}^{k} q^{-\frac{1}{2}l(l+2)+s(k+s-2)} \begin{bmatrix} k \\ s \end{bmatrix}_q [[\beta^{j+k-s}, \alpha^s], e_{n+l+2s}], \end{aligned} \tag{4.18}$$

completing the calculation of the action of T_β on $A_r(X_r, X_r^0; L_r)$.

4.4. Action of $\mathcal{M}_{1,1}(X)$ on $U_q^{red}(sl_2(\mathbb{C}))$

We have identified $A_r(X_r, X_r^0; L_r)$ as a $U_q^{red}(sl_2(\mathbb{C}))$ module as $U_q^{red}(sl_2(\mathbb{C}))$ with the adjoint action. The action of $\mathcal{M}_{1,1}(X)$ on $A_r(X_r, X_r^0; L_r)$ commutes with the topological action of $U_q^{red}(sl_2(\mathbb{C}))$. In this section, we identify the action of $\mathcal{M}_{1,1}(X)$ on $U_q^{red}(sl_2(\mathbb{C}))$ defined by its action on $A_r(X_r, X_r^0; L_r)$. By construction it commutes with the adjoint action.

Action of T_α

Let $\phi_r : A_r(X_r, X_r^0; L_r) \to U_q^{red}(sl_2(\mathbb{C}))$ be the restriction of the map defined in chapter 3. Then

$$\begin{aligned} &\phi_r\left\{L_r(T_\alpha)([[\beta^j, \alpha^k], e_n])\right\} \\ &\quad = dq^{-\frac{1}{2}n^2} \sum_{s=0}^{\min\{j,p-k+1\}} q^{s(j+2k+s-2)} \begin{bmatrix} j \\ s \end{bmatrix}_q \\ &\qquad \times N(j-s, k+s, n+2s) F^{k+s} T_{n+2s-1} E^{p-j+s-1}, \end{aligned} \tag{4.19}$$

using (4.13). We define $U(T_\alpha) : U_q^{red}(sl_2(\mathbb{C})) \to U_q^{red}(sl_2(\mathbb{C}))$ by

$$U(T_\alpha) \circ \phi_r = \phi_r \circ L_r(T_\alpha). \tag{4.20}$$

Thus, what is left to compute $U(T_\alpha)$, is the ratio of normalization constants

$$\frac{N(j-s, k+s, n+2s)}{N(j,k,n)} = (-1)^s q^{-s(j+2k+\frac{3}{2}s+n-\frac{3}{2})}[1]_q^s \frac{[j-s]_q!}{[j]_q!}. \tag{4.21}$$

We conclude that

$$\begin{aligned} &U(T_\alpha)(F^k T_{n-1} E^{p-j-1}) \\ &\quad = dq^{-\frac{1}{2}n^2} \sum_{s=0}^{\min\{j,p-1-k\}} (-1)^s q^{-\frac{1}{2}s(s+1)-sn} \frac{[1]_q^s}{[s]_q!} F^{k+s} T_{n+2s-1} E^{p-j+s-1}, \end{aligned} \tag{4.22}$$

giving the action of $U(T_\alpha)$ on $U_q^{red}(sl_2(\mathbb{C}))$.

Action of T_β

Let $r = j+k$. Using (4.18), we obtain

$$\begin{aligned} &\phi_r \left\{ L_r(T_\beta)([[\beta^j, \alpha^k], e_n]) \right\} \\ &\quad = \sum_{l=0}^{2p-1} \sum_{s=\max\{0,j+k-p+1\}}^{k} q^{-\frac{1}{2}l(l+2)+s(k+s-2)} \begin{bmatrix} k \\ s \end{bmatrix}_q \\ &\qquad \times N(j+k-s, s, n+l+2s) F^s T_{n+l+2s-1} E^{p-j-k+s-1}. \end{aligned} \tag{4.23}$$

Insert

$$\begin{aligned} &\frac{N(j+k-s, s, n+l+2s)}{N(j,k,n)} \\ &\quad = \frac{(-1)^{k-s}}{[1]_q^{k-s}} q^{\frac{1}{2}(k-s)(k-s+1)+(k-s)(j+n+2)+(2s+l)(2-s)} \frac{[j+k-s]_q!}{[j]_q!} \end{aligned} \tag{4.24}$$

to conclude that

$$\begin{aligned} &U(T_\beta)(F^k T_{n-1} E^{p-j-1}) \\ &\quad = \sum_{l=0}^{2p-1} \sum_{s=\max\{0,j+k-p+1\}}^{k} \frac{(-1)^{k-s}}{[1]_q^{k-s}} q^{\frac{1}{2}(k-s)(k-s+1)+(k-s)(j+n+s+2)+\frac{1}{2}(l+2s)(1-l)} \\ &\qquad \times [k-s]_q! \begin{bmatrix} j+k-s \\ j \end{bmatrix}_q \begin{bmatrix} k \\ s \end{bmatrix}_q F^s T_{n+l+2s-1} E^{p-j-k+s-1} \end{aligned} \tag{4.25}$$

completing the calculation of $U(T_\beta)$.

4.5. Action of $S_{\alpha\beta}$

We define $S_{\alpha\beta} := T_\beta(T_\alpha)^{-1}T_\beta$. The action of $S_{\alpha\beta}$ on $\pi_1(X, p_0)$ is given by $S_{\alpha\beta}(\alpha) = \beta$ and $S_{\alpha\beta}(\beta) = \alpha'$. Recall that $\alpha' = \beta^{-1} \circ \alpha^{-1} \circ \beta$. That is $S_{\alpha\beta}$ maps α to β and β to α^{-1} conjugated by β^{-1}. This transformation is known as the S-transformation. We put

$$\begin{aligned} D(S_{\alpha\beta}) &:= D(T_\beta)D(T_\alpha)^{-1}D(T_\beta), \\ D(S_{\alpha\beta})_{m,n} &= \frac{2pq}{d^2} q^{(m+1)(n-1)}. \end{aligned} \tag{4.26}$$

$D(S_{\alpha\beta})$ performs a discrete Fourier transformation on V. $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$ is compatible with $S_{\alpha\beta}$, the equivalence being given by $D(S_{\alpha\beta})$. We thus have an action $L_r(S_{\alpha\beta})$ on $A_r(X_r, X_r^0; L_r)$. It has the form

$$L_r(S_{\alpha\beta})([[\beta^j, \alpha^k], e_n]) = [[\alpha'^j, \beta^k], D(S_{\alpha\beta})e_n], \tag{4.27}$$

$r = j + k$. As before, we deduce from (4.27) an action $U(S_{\alpha\beta})$ on $U_q^{red}(sl_2(\mathbb{C}))$. However, we do not expand (4.27) in terms of the basis of $A_r(X_r, X_r^0; L_r)$ as we did in the case of $L_r(T_\alpha)$ and $L_r(T_\beta)$, although this could be done. Instead we compute directly the image of (4.27) under ϕ_r. Using

$$[[\alpha'^j, \beta^k], e_n] = q^{j(j+2k+1)}(F_R)^j[[\beta^k], e_n] \tag{4.28}$$

it follows that

$$\begin{aligned} &\phi_r([[\alpha'^j, \beta^k], e_n]) \\ &\quad = (-1)^k [1]_q^{j-k} \frac{[k]_q!}{[j]_q!} q^{\frac{1}{2}j(j+1)+\frac{1}{2}k(k+1)+2k(j+1)+n(j+k)} \\ &\quad \times N(j,k,n) T_{n-1} E^{p-k-1} F^j. \end{aligned} \tag{4.29}$$

Note that by reexpressing (4.29) in terms of the basis $F^jT_{k-1}E^{p-l-1}$ and applying $(\phi_r)^{-1}$, the expansion of (4.27) in terms of the basis of $A_r(X_r, X_r^0; L_r)$ could be obtained. We conclude that

$$\begin{aligned} &\phi_r \left\{ L_r(S_{\alpha\beta})([[\beta^j, \alpha^k], e_n]) \right\} \\ &\quad = \frac{2pq}{d^2} (-1)^k [1]_q^{j-k} \frac{[k]_q!}{[j]_q!} q^{\frac{1}{2}j(j+3)+\frac{1}{2}k(k+3)+k(n+1+2j)} \\ &\quad \times N(j,k,n) K^{2(j+n+1)} E^{p-k-1} F^j, \end{aligned} \tag{4.30}$$

using

$$\sum_{l=0}^{2p-1} q^{(n+l-1)(j+n-1)} T_{n+l-1} = K^{2(j+n+1)}. \tag{4.31}$$

The final result is

$$\begin{aligned} &U(S_{\alpha\beta})(F^k T_{n-1} E^{p-j-1}) \\ &\quad = \frac{2pq}{d^2}(-1)^k [1]_q^{j-k} \frac{[k]_q!}{[j]_q!} q^{\frac{1}{2}j(j+3)+\frac{1}{2}k(k+3)+k(n+1+2j)} K^{2(j+n+1)} E^{p-k-1} F^j. \end{aligned} \tag{4.32}$$

$U(S_{\alpha\beta})$ is the algebraic version of the S-transformation. It is a mapping of $U_q^{red}(sl_2(\mathbb{C}))$ to itself, one-to-one and onto, which commutes with the adjoint action.

5. Identification of the S- and the T-transformation

We identify the operations $U(T_\alpha)$ and $U(S_{\alpha\beta})$ in terms of the quasitriangular structure of $U_q^{red}(sl_2(\mathbb{C}))$. Let us first adjust the normalization of $D(T_\alpha)$ and $D(S_{\alpha\beta})$ as follows:

$$\begin{aligned} D(T_\alpha) &\to \frac{N_\alpha}{d} D(T_\alpha), \\ D(S_{\alpha\beta}) &\to \frac{d^2 N_{\alpha\beta}}{2pq} D(S_{\alpha\beta}). \end{aligned} \tag{5.1}$$

With this change of normalization, $U(T_\alpha)$ and $U(S_{\alpha\beta})$ act on $U_q^{red}(sl_2(\mathbb{C}))$ by

$$\begin{aligned} &U(T_\alpha)(F^k T_{n-1} E^{p-j-1}) \\ &\quad = N_\alpha q^{-\frac{1}{2}n^2} \sum_{s=0}^{\min\{j,p-1-k\}} (-1)^s q^{-\frac{1}{2}s(s+1)-sn} \frac{[1]_q^s}{[s]_q!} F^{k+s} T_{n+2s-1} E^{p-j+s-1}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &U(S_{\alpha\beta})(F^k T_{n-1} E^{p-j-1}) \\ &\quad = N_{\alpha\beta}(-1)^k [1]_q^{j-k} \frac{[k]_q!}{[j]_q!} q^{\frac{1}{2}j(j+3)+\frac{1}{2}k(k+3)+k(n+1+2j)} K^{2(j+n+1)} E^{p-k-1} F^j. \end{aligned} \tag{5.3}$$

5.1. Universal elements of $U_{q,K}^{red}(sl_2(\mathbb{C}))$

Let us consider the K generated version of $U_q^{red}(sl_2(\mathbb{C}))$. It is known to be a ribbon Hopf algebra. The universal R-matrix is

$$R = \frac{1}{4p}\left(\sum_{n=0}^{p-1}(-1)^n\frac{[1]_q^n}{[n]_q!}q^{-\frac{1}{2}n(n-1)}E^n\otimes F^n\right)\left(\sum_{m,n=0}^{4p-1}q^{\frac{1}{2}nm}K^n\otimes K^m\right). \tag{5.4}$$

The associated central element V is

$$\begin{aligned} V &= \sum_{n=0}^{p-1}\sum_{m=0}^{4p-1}(-1)^n\frac{[1]_q^n}{[n]_q!}q^{\frac{1}{2}n(n+1)+n(m+1)+\frac{1}{2}m(m+2)}F^nH_{m+2n}E^n, \\ H_n &= \frac{1}{4p}\sum_{m=0}^{4p-1}q^{-\frac{1}{2}nm}K^m. \end{aligned} \tag{5.5}$$

5.2. Identification of $U(T_\alpha)$

Let $N_\alpha = q^{\frac{1}{2}}$. Then

$$\begin{array}{ccc} U_q^{red}(sl_2(\mathbb{C})) & \xrightarrow{U(T_\alpha)} & U_q^{red}(sl_2(\mathbb{C})) \\ \downarrow & & \downarrow \\ U_{q,K}^{red}(sl_2(\mathbb{C})) & \xrightarrow{\lambda_{V^{-1}}} & U_{q,K}^{red}(sl_2(\mathbb{C})) \end{array} \tag{5.6}$$

commutes. That is, $U(T_\alpha)$ is identified with the multiplication by the inverse of the central element V of (5.4). This is shown by

$$\begin{aligned} &V^{-1}F^kT_{n-1}E^{p-j-1} \\ &\quad= \sum_{s=0}^{p-1}\sum_{l=0}^{4p-1}(-1)^s\frac{[1]_q^s}{[s]_q!}q^{-\frac{1}{2}s(s+1)-sl-\frac{1}{2}l^2+\frac{1}{2}}F^{k+s}H_{l+2s-1}T_{n+2s-1}E^{p-1-j+s} \end{aligned} \tag{5.7}$$

with

$$\begin{aligned} H_{l+2s-1}T_{n+2s-1} &= H_{l+2s-1}(H_{n+2s-1}+H_{n+2p+2s-1}) \\ &= \delta_{l,n}H_{n+2s-1}+\delta_{l,n+2p}H_{n+2p+2s-1}, \end{aligned} \tag{5.8}$$

comparing the result with (5.2).

5.3. Trace on $U_{q,K}^{red}(sl_2(\mathbb{C}))$

Let $\tau : U_{q,K}^{red}(sl_2(\mathbb{C})) \to \mathbb{C}$ be the linear map such that

$$\begin{aligned} \tau(E^{p-1}F^{p-1}H_{n-1}) &:= 1, \qquad 1\le n\le 4p, \\ \tau(E^jF^kH_{n-1}) &:= 0, \qquad \text{else}. \end{aligned} \tag{5.9}$$

τ is a trace on $U_{q,K}^{red}(sl_2(\mathbb{C}))$: For $X = E, F, K^{\pm 1}$, and $Y \in U_{q,K}^{red}(sl_2(\mathbb{C}))$, $\tau(XY) = \tau(YX)$.

5.4. S-transformation

Let $R = \sum_i \alpha_i \otimes \beta_i$ the universal R-matrix (5.4). We define a linear map $S : U_{q,K}^{red}(sl_2(\mathbb{C})) \to U_{q,K}^{red}(sl_2(\mathbb{C}))$ by

$$S(X) := \sum_{j,l} \beta_j \eta(\alpha_l) \tau(\eta(\beta_l) K^{-2} \alpha_j X) \tag{5.10}$$

with τ the trace of (5.9). A short computation reveals that

$$\begin{aligned} &S(T_{n-1} E^{p-1-r} F^{p-1-s}) \\ &\quad = \frac{2(-1)^{r+s}}{[r]_q![s]_q!} q^{-\frac{1}{2}r(r+1) - \frac{1}{2}s(s+1) + (s-1)(2r+n-1)} F^r K^{2(1-n-r)} E^s. \end{aligned} \tag{5.11}$$

We thus obtain a map $S : U_q^{red}(sl_2(\mathbb{C})) \to U_q^{red}(sl_2(\mathbb{C}))$ by restriction.

5.5. Identification of $S_{\alpha\beta}$

Put the normalization constant in (5.1) to be

$$N_{\alpha\beta} := \frac{1}{2p} \frac{[p-1]_q!}{[1]_q^{p-1}} q^{-\frac{1}{2}(p-1)(p+2)}. \tag{5.12}$$

The transformation (5.3) is identified as

$$U(S_{\alpha\beta})^{-1} X = S(X) \tag{5.13}$$

with S the transformation (5.11). To verify (5.13), we compute the inverse transformation of (5.3). it is seen to be

$$\begin{aligned} &U(S_{\alpha\beta})^{-1}(T_{n-1} E^{p-1-k} F^j) \\ &\quad = \frac{1}{pN_{\alpha\beta}} (-1)^k [1]_q^{k-j} \frac{[j]_q!}{[k]_q!} q^{-\frac{1}{2}j(j+3) - \frac{1}{2}k(k+3) - jk - (n+k-1)(j+2)} F^k K^{-2(n+k-1)} E^{p-j-} \end{aligned} \tag{5.14}$$

Setting $k = r$ and $j = p-1-s$ and comparing (5.11) with (5.14), the result (5.13) follows.

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